J. Fluid Mech. (1970), *vol.* 43, *part* 1, *pp.* 129–144
Printed in Great Britain

The effect of weak stratification and geometry on the steady motion of a contained rotating fluid

By J. S. ALLEN

Department of Aerospace Engineering,
The Pennsylvania State University

(Received 10 October 1969)

The effect of a density stratification on the steady, mechanically driven motion of a viscous fluid in a rotating cylinder with axis aligned with the rotation and gravity vectors and with parallel top and bottom surfaces that slope with respect to the plane perpendicular to the rotation vector is studied by a linear theory. Primary attention is given to a study of the alteration of the characteristics of the flow of a homogeneous fluid by the addition of a weak stratification. It is found, for example, that in the range $E^{\frac{3}{2}} < \sigma S < E$, where $E = \nu/\Omega L^2$ and $\sigma S = \nu\alpha g \Delta T_0/\kappa\Omega^2 L$, and with a homogeneous boundary condition on the perturbation temperature, the interior velocity is parallel to the direction perpendicular to the plane determined by the vector normal to the top surface and the rotation vector. The circulation closes in an inviscid, but heat-conducting, boundary layer of thickness $E^{\frac{3}{4}}(\sigma S)^{-\frac{1}{2}}$ on the side wall. Thus, with stratification, the steady flow in this configuration differs markedly from the corresponding flow in a cylinder where the top and bottom surfaces lie in planes perpendicular to the rotation vector. The difference is caused by the fact that in the container with sloping surfaces the basic stratification interacts with the geostrophic flow whereas, in the other case, the interaction is with the much smaller Ekman layer suction velocities.

1. Introduction

The effect of a stable density stratification on the steady, linear motion of a contained rotating fluid has been studied by Barcilon & Pedlosky (1967*a*). In a second paper (1967*b*) they developed a unified linear theory which showed how, as the basic stratification was increased, the results from the theory of homogeneous, rotating fluids merged with the results for substantial stratifications. In the latter case, they considered the steady, mechanically driven motion in a cylindrical container with a boundary condition of zero heat flux on the perturbation temperature. The rotation and gravity vectors were assumed to be antiparallel and the axis of the cylinder was aligned with the rotation vector. The top and bottom surfaces of the container were flat; that is, they were formed by parallel planes that were perpendicular to the rotation vector. In that problem the primary interaction of the flow with the basic stratification was through the small $O(E^{\frac{1}{2}})$, where E is the Ekman number, vertical velocities pumped by the Ekman layers on the top and bottom surfaces.

Now if we consider, for a *homogeneous* fluid, the steady, mechanically driven, linear motion in a rotating cylinder with top and bottom surfaces parallel, but sloping with a small angle with respect to the plane perpendicular to the rotation vector (see figure 1), we find that the resulting flow is qualitatively the same as that in the corresponding problem in a cylinder with flat top and bottom surfaces. Under the restrictions of the linear theory the fluid can move in a similar manner in both containers since the distance parallel to the rotation vector, between the top and bottom surfaces, does not vary in either case. If, however, we consider the changes in the characteristics of the flow as a weak stratification is added and increased, we find that the results for the two geometries differ considerably. The essential difference is caused by the fact that in the cylinder with sloping surfaces the stratification interacts with the order one geostrophic velocity, whereas in the cylinder with flat surfaces the primary interaction is with the much smaller Ekman-layer suction velocities. For convenience we will refer to the container geometry shown in figure 1 as the 'doubly sliced' cylinder.

The steady flow in the 'doubly sliced' cylinder is of interest, therefore, because it presents a model where the effects of rotation and stratification interact in a fundamental manner which is basically different from that studied previously. We remark that some aspects of the *unsteady* inviscid motion of a weakly stratified, rotating fluid in containers with geometries similar to that of the 'doubly sliced' cylinder have been investigated (Allen 1968) and the appearance of some rather interesting low-frequency wave motions has been predicted.

In the following sections we present a study of the effect of a weak stratification on the steady motion of a rotating fluid in the 'doubly sliced' cylinder geometry.

2. Formulation

We consider a viscous, heat-conducting, incompressible fluid, which satisfies the Boussinesq approximation, in a frame of reference rotating with a uniform angular velocity $\mathbf{\Omega} = \Omega\hat{\mathbf{k}}$ and acted on by a gravitational acceleration $\mathbf{g} = -g\hat{\mathbf{k}}$ which is antiparallel to the rotation vector. The governing equations for steady motion are

$$\nabla\cdot\mathbf{q} = 0,$$

$$\mathbf{q}\cdot\nabla\mathbf{q} + 2\Omega\hat{\mathbf{k}}\times\mathbf{q} = -(1/\rho_0)\nabla p - (\rho/\rho_0)\,g\hat{\mathbf{k}} + \tfrac{1}{2}(\rho/\rho_0)\,\Omega^2\nabla|\hat{\mathbf{k}}\times\mathbf{r}|^2 + \nu\nabla^2\mathbf{q},$$

$$\mathbf{q}\cdot\nabla T = \kappa\nabla^2 T,$$

$$\rho = \rho_0[1-\alpha(T-T_0)],$$

where $\mathbf{q}$, p, ρ and T are respectively the velocity, pressure, density and temperature of the fluid at a point $\mathbf{r}$; ν, κ and α are respectively the constant kinematic viscosity, thermometric conductivity, and coefficient of thermal expansion; ρ_0 and T_0 are constant reference values of the density and temperature.

We assume that the Froude number $\Omega^2 L/g$ is small and consider a linear equilibrium temperature and density distribution (see Greenspan 1968, §1.4) given by

$$T_s = T_0 + \Delta T_0 z/L,$$

$$\rho_s = \rho_0[1-\alpha\Delta T_0 z/L],$$

where ΔT_0 is the basic temperature difference imposed over the height L.

The variables are non-dimensionalized in the following manner:

$$\mathbf{q} = U\mathbf{q}^*, \quad \mathbf{r} = L\mathbf{r}^*, \quad t = \Omega^{-1}t^*,$$

$$p = p_0 - \rho_0 gLz^* + \tfrac{1}{2}\rho_0 gL\alpha\Delta T_0 z^{*2} + \rho_0 U\Omega Lp^*,$$

$$T = T_s + \frac{\Omega U}{\alpha g} T^*, \quad \rho = \rho_s + \frac{\rho_0 U\Omega}{g}\rho^*,$$

where p_0 is a constant reference pressure and U is a reference velocity.

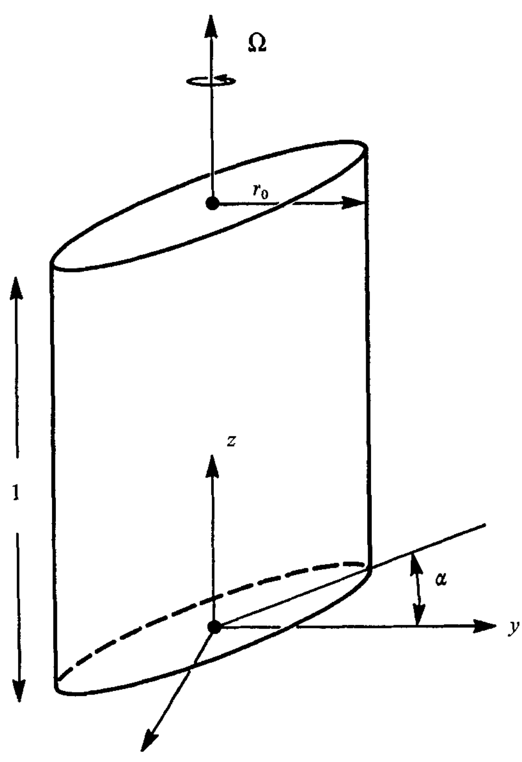

FIGURE 1. The 'doubly sliced' cylinder configuration where the angle $\alpha = \tan^{-1}(b/a)$.

Assuming that the Rossby number $\epsilon = U/\Omega L$ is small such that the non-linear terms multiplied by ϵ can be neglected, we find that the resulting dimensionless equations are (dropping the asterisks)

$$\nabla \cdot \mathbf{q} = 0, \tag{2.1}$$

$$2\hat{\mathbf{k}} \times \mathbf{q} = -\nabla p + T\hat{\mathbf{k}} + E\nabla^2\mathbf{q}, \tag{2.2}$$

$$\sigma S\mathbf{q} \cdot \hat{\mathbf{k}} = E\nabla^2 T, \tag{2.3}$$

where $E = \nu/\Omega L^2$ is the Ekman number, $\sigma = \nu/\kappa$ is the Prandtl number, and $S = \alpha g\Delta T_0/\Omega^2 L$ is the inverse of the internal Froude number. The parameter S is a measure of the stratification and can also be written as $S = N^2/\Omega^2$, where N is the Brunt–Väisälä frequency.

The cylindrical container is shown in figure 1. The z axis is aligned with the axis of the cylinder and with the basic rotation vector. The top and bottom surfaces are parallel planes with the outward pointing unit normal vectors given by

$$\hat{\mathbf{n}}_T = -\hat{\mathbf{n}}_B = a\hat{\mathbf{k}} - b\hat{\mathbf{j}},$$

where a and b are the direction cosines, such that $a^2+b^2 = 1$, and $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are the unit vectors in the x, y and z directions. The dimensionless radius and height are r_0 and 1 respectively.

It will be convenient to refer to the velocity components in both Cartesian and cylindrical polar co-ordinates and we use the following notation:

$$\mathbf{q} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}} = u\hat{\mathbf{r}} + v\hat{\boldsymbol{\theta}} + w\hat{\mathbf{k}},$$

where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are unit vectors in the r and θ directions and the line $\theta = 0$ is aligned with the positive x axis.

We consider the motion driven by a rigid rotation of the top surface in its own plane with angular velocity, relative to the rotating frame, of $\epsilon r_0^{-1}\Omega$ which in dimensionless form we write as $\omega_T = r_0^{-1}$. The boundary conditions on the velocity are therefore

$$\mathbf{q} = \omega_T\{x\hat{\mathbf{i}} + [ay + b(z-1)][a\hat{\mathbf{j}} + b\hat{\mathbf{k}}]\} \quad \text{on} \quad z = z_T = 1 + (b/a)\,y$$

and $$\mathbf{q} = 0 \quad \text{on} \quad z = z_B = (b/a)\,y \quad \text{and on} \quad r = r_0.$$

We use either $T = 0$ or $\hat{\mathbf{n}}\cdot\nabla T = 0$ as the boundary condition for the temperature field.

The Ekman number E will be assumed to be small and boundary-layer methods for the limit $E \to 0$ will be used. The parameters σ and S appear in the combination σS which, following Barcilon & Pedlosky (1967*b*), will be called the stratification. Note that for $\sigma S = 0$ the temperature is equal to a constant and, if the pressure is redefined, the equations reduce to those governing the motion of a homogeneous fluid. Since we are interested in the modifications of the characteristics of the flow of a homogeneous fluid produced by the addition of a weak stratification, σS will also be a small parameter. As usual, to maintain the validity of the approximate solutions, it is necessary to relate the two small parameters σS and E and, in general, σS will be ordered with respect to some power of E as $E \to 0$. In addition, to facilitate solving the problem, we will consider that the top and bottom surfaces have a small angle of slope. That is, b will also be considered a small parameter. However, in general, b will be considered to be of order one compared with σS or E as $E \to 0$. Because of the attendant difficulties involved in the asymptotic solution of a problem with three small parameters we will only attempt to find the lowest order approximations for the flow variables.

In some of the following arguments, concerning the inviscid interior flow, it is convenient to use the 'thermal wind' relation

$$\hat{\mathbf{k}}\,\hat{\mathbf{k}}\cdot\nabla\mathbf{q} = \tfrac{1}{2}\times\nabla T, \tag{2.4}$$

obtained by taking the curl of equation (2.2), using (2.1), and neglecting the viscous term. In particular, the y component of this equation

$$v_z = \tfrac{1}{2}T_x, \tag{2.5}$$

where the subscripts denote partial differentiation, will be useful.

Some of the consequences of stratification in the geometry of the 'doubly sliced' cylinder can be seen from the following order-of-magnitude considerations. For small values of the stratification the flow in the interior is expected to be invisicid and to have an order 1 velocity that satisfies $\mathbf{q}\cdot\hat{\mathbf{n}} = 0$ on the boundary. For example, on the top surface we have

$$\mathbf{q}\cdot\hat{\mathbf{n}}_T = aw - bv = 0. \tag{2.6}$$

It follows that both w and v will be of order 1. Now with $w = O(1)$ we see from the energy equation that $T = O(\sigma S/E)$. Consequently, when σS increases to the point where $\sigma S = O(E)$, we find that T will be $O(1)$ and will enter into the thermal wind equation (2.4) for the order one velocity field. Therefore we can expect $\sigma S = O(E)$ to be a critical stratification.

Now let us consider the case where $\sigma S > O(E)$. From (2.3) we have $w = O(E/\sigma S)\,T$. Surmising from (2.5) that, for an order 1 temperature field, v and T in general have the same order of magnitude, we find $w = O(E/\sigma S)\,v$. So, for $\sigma S > O(E)$, we can conclude that v should be of a larger order than w. However, at the same time the boundary condition (2.6) implies that v and w must be the *same* order, which obviously results in a contradiction. Note that in these arguments there has been nothing said about the velocity in the x direction since $\hat{\mathbf{n}}_T\cdot\hat{\mathbf{i}} = 0$. The above contradiction implies that, for $\sigma S > O(E)$, the flow must adjust in a manner which differs from that implicitly assumed in these rough order-of-magnitude arguments. In fact, we will see in the next section that, for increasing σS, the characteristics of the flow change drastically *before* the critical stratification $\sigma S = O(E)$ is reached.

3. Analysis

We start by examining the effect of the smallest stratification that can substantially alter the motion. This value is $\sigma S = O(E^{\frac{3}{2}})$. In this case the temperature field that is generated by the vertical velocity does not affect the order 1 motion directly. However, it does affect the $O(E^{\frac{1}{2}})$ flow induced by the Ekman layers and this in turn can alter the lowest-order flow field. Therefore, we set $\lambda = \sigma S/E^{\frac{3}{2}}$ and, for the interior, expand the variables in powers of $E^{\frac{1}{2}}$.

$$\mathbf{q} = \mathbf{q}_0 + E^{\frac{1}{2}}\mathbf{q}_1 + \ldots,$$

$$p = p_0 + E^{\frac{1}{2}}p_1 + \ldots,$$

$$T = E^{\frac{1}{2}}T_0 + \ldots.$$

Substituting the expansions into (2.1)–(2.3) we find that the $O(1)$ and $O(E^{\frac{1}{2}})$ interior equations are

$$\nabla\cdot\mathbf{q}_0 = 0, \tag{3.1}$$

$$2\hat{\mathbf{k}}\times\mathbf{q}_0 = -\nabla p_0, \tag{3.2}$$

$$\nabla\cdot\mathbf{q}_1 = 0, \tag{3.3}$$

$$2\hat{\mathbf{k}}\times\mathbf{q}_1 = -\nabla p_1 + T_0\hat{\mathbf{k}}, \tag{3.4}$$

$$\lambda\mathbf{q}_0\cdot\hat{\mathbf{k}} = \nabla^2 T_0. \tag{3.5}$$

The $O(1)$ equations are the same as those for the geostrophic flow of a homogeneous fluid. From (3.1) and (3.2) we can obtain the Taylor–Proudman theorem

$$\hat{\mathbf{k}}\cdot\nabla\mathbf{q}_0 = \hat{\mathbf{k}}\cdot\nabla p_0 = 0. \tag{3.6}$$

With the use of the inviscid boundary condition $\mathbf{q}_0\cdot\hat{\mathbf{n}} = 0$, applied on the top and bottom surfaces, the velocity field can be expressed as

$$\mathbf{q}_0 = \tfrac{1}{2}a^{-1}\hat{\mathbf{n}}_T\times\nabla p_0. \tag{3.7}$$

The boundary condition $\mathbf{q}_0\cdot\hat{\mathbf{r}} = 0$ on the side wall must also be satisfied and in terms of the pressure this becomes $p_{0\theta} = 0$ or preferably

$$p_0 = 0 \quad \text{on} \quad r = r_0. \tag{3.8}$$

The $O(E^{\frac{1}{2}})$ variables are conveniently expressed in terms of their components in Cartesian co-ordinates,

$$p_1 = \int_{z_B}^{z} T_0\,dz + C_1(x,y), \tag{3.9}$$

$$v_1 = \frac{1}{2}\frac{\partial p_1}{\partial x} = \frac{1}{2}\frac{\partial}{\partial x}\int_{z_B}^{z} T_0\,dz + \frac{1}{2}\frac{\partial C_1}{\partial x}, \tag{3.10}$$

$$u_1 = -\frac{1}{2}\frac{\partial p_1}{\partial y} = -\frac{1}{2}\frac{\partial}{\partial y}\int_{z_B}^{z} T_0\,dz - \frac{1}{2}\frac{\partial C_1}{\partial y}, \tag{3.11}$$

where C_1 is an undetermined function of x and y. In addition, from (3.3) and (3.4) we find

$$\partial w_1/\partial z = 0. \tag{3.12}$$

At these low values of the stratification the boundary layers on the top and bottom surfaces are ordinary Ekman layers and the expressions for the $O(E^{\frac{1}{2}})$ normal fluxes reduce to (see Greenspan 1968, §2.6)

$$\mathbf{q}_1\cdot\hat{\mathbf{n}}_T = -\tfrac{1}{2}a^{-\frac{1}{2}}(\boldsymbol{\zeta}_0-\boldsymbol{\zeta}_T)\cdot\hat{\mathbf{n}}_T \quad \text{at} \quad z = z_T = 1+(b/a)\,y, \tag{3.13}$$

$$\mathbf{q}_1\cdot\hat{\mathbf{n}}_B = \tfrac{1}{2}a^{-\frac{1}{2}}\boldsymbol{\zeta}_0\cdot\hat{\mathbf{n}}_B \quad \text{at} \quad z = z_B = (b/a)\,y, \tag{3.14}$$

where $\boldsymbol{\zeta}_0 = \nabla\times\mathbf{q}_0$ is the vorticity and $\boldsymbol{\zeta}_T\cdot\hat{\mathbf{n}}_T = 2\omega_T$ is twice the angular velocity of the rigidly rotating top surface.

The $O(1)$ flow field is determined by utilizing the equations (3.13) and (3.14) for the normal velocities induced by the Ekman layers. These equations can be written

$$aw_1 - bv_1(z = z_T) = -\tfrac{1}{4}a^{-\frac{3}{2}}\nabla_T^2 p_0 + a^{-\frac{1}{2}}\omega_T \tag{3.15}$$

and

$$aw_1 - bv_1(z = z_B) = +\tfrac{1}{4}a^{-\frac{3}{2}}\nabla_T^2 p_0, \tag{3.16}$$

where we have substituted equation (3.7) for $\mathbf{q}_0$ and where only the explicit dependence of v_1 on z is indicated. Also, we use the notation that

$$\nabla_T^2 = \frac{\partial^2}{\partial x^2} + (1-b^2)\frac{\partial^2}{\partial y^2}.$$

Subtracting (3.16) from (3.15), and using the z independence of p_0 and w_1 from (3.6) and (3.12) we find, after substituting expression (3.10), that

$$\frac{\partial^2 p_0}{\partial x^2} + (1-b^2)\frac{\partial^2 p_0}{\partial y^2} = a^{\frac{3}{2}}b\,\frac{\partial}{\partial x}\int_{z_B}^{z_T} T_0\,dz + 2a\omega_T. \tag{3.17}$$

A second equation relating p_0 and T_0 is given by (3.5), which can be written

$$\frac{\partial^2 T_0}{\partial x^2}+\frac{\partial^2 T_0}{\partial y^2}+\frac{\partial^2 T_0}{\partial z^2}=\frac{\lambda b}{2a}\frac{\partial p_0}{\partial x}, \tag{3.18}$$

where we have used $\mathbf{q}_0 \cdot \hat{\mathbf{n}}_T = 0$, (3.6) and (3.2) to write

$$w_0 = (b/a)\, v_0 = (b/2a)\, p_{0x}.$$

Equations (3.17) and (3.18), for the two variables p_0 and T_0, have to be solved simultaneously with the boundary condition (3.8) on the pressure and, since no viscous boundary layer can affect the temperature or heat flux at these low values of the stratification, with either $T = 0$ or $\hat{\mathbf{n}} \cdot \nabla T = 0$ as the boundary condition on the temperature.

As we mentioned before, it is convenient to find an approximate solution to this problem for small slopes of the top and bottom surfaces; that is, for small values of the parameter b. This allows us to make certain simplifications in (3.17) and (3.18). We are, however, mainly interested in the behaviour of the solutions as λ attains large values and consequently we have to retain certain of the terms multiplied by b. Therefore, considering $b \ll 1$ and retaining terms that are important for $b\lambda^{\frac{1}{2}} \gg 1$, we find that (3.17) and (3.18) can be approximated by

$$\frac{\partial^2 p_0}{\partial x^2}+\frac{\partial^2 p_0}{\partial y^2}=b\frac{\partial}{\partial x}\int_0^1 T_0\, dz+2\omega_T, \tag{3.19}$$

and

$$\frac{\partial^2 T_0}{\partial x^2}+\frac{\partial^2 T_0}{\partial y^2}+\frac{\partial^2 T_0}{\partial z^2}=\frac{\lambda b}{2}\frac{\partial p_0}{\partial x}, \tag{3.20}$$

where the boundary conditions on the top and bottom surfaces are now to be applied at $z = 1$ and $z = 0$, respectively, and where, when the heat flux condition is used on these surfaces it becomes, in the first approximation, $T_{0z} = 0$.

Let us first examine the solutions for $\lambda = 0$. In that case it follows that T_0 = constant, whereupon (3.19) reduces to

$$\frac{\partial^2 p_0}{\partial x^2}+\frac{\partial^2 p_0}{\partial y^2}=2\omega_T. \tag{3.21}$$

The solution, with boundary condition (3.8), is

$$p_0 = \tfrac{1}{2}\omega_T(r^2-r_0^2),$$

which gives

$$v_0 = \tfrac{1}{2}(\partial p_0/\partial r) = \tfrac{1}{2}\omega_T r. \tag{3.22}$$

Thus, the first approximation for $b \ll 1$ and $\lambda = 0$ is identical to the corresponding result for a homogeneous fluid in a cylinder with flat top and bottom surfaces. In addition, there would, of course, be viscous boundary layers of thickness $E^{\frac{1}{4}}$ and $E^{\frac{1}{3}}$ on the side walls but these do not affect the interior dynamics.

In order to see clearly the effect on the flow field of an increased stratification it is appropriate to consider large values of λ and to solve (3.19) and (3.20) asymptotically for $\lambda \gg 1$. Once the solution is obtained we can check on just how large λ can be before the approximations made in arriving at the solutions are invalidated. We remark that similar equations, without the z dependence, arise in magnetohydrodynamic pipe flow. Some cases are solved by Cole (1968, §4.3), where references to the literature are given.

Considering equations (3.19) and (3.20) for $\lambda \gg 1$ we find that the solutions exhibit a boundary-layer behaviour, with boundary layers of thickness $\lambda^{-\frac{1}{2}}$ at the side walls. We therefore assume a composite expansion, the appropriate form of which is

$$T_0 = T_{00} + \lambda^{-\frac{1}{2}} T_{01} + \ldots + \tilde{T}_{00} + \lambda^{-\frac{1}{2}} \tilde{T}_{01} + \ldots, \tag{3.23}$$

$$p_0 = \lambda^{-\frac{1}{2}} (p_{00} + \lambda^{-\frac{1}{2}} p_{01} + \ldots + \tilde{p}_{00} + \lambda^{-\frac{1}{2}} \tilde{p}_{01} + \ldots), \tag{3.24}$$

where the tilde symbol denotes a boundary-layer function of the boundary-layer variable $\rho = (r_0 - r)\lambda^{\frac{1}{2}}$. These functions approach zero exponentially fast as $\rho \to \infty$. We also separate out the z averaged part of the temperature by writing

$$T_0 = \langle T_0 \rangle + \mathscr{T}_0,$$

where

$$\langle T_0 \rangle = \int_0^1 T_0 \, dz.$$

Substituting the expansions into the equations we find that the lowest order interior equations are

$$\frac{\partial}{\partial x} \langle T_{00} \rangle = -2\omega_T b^{-1}, \tag{3.25}$$

$$\frac{\partial}{\partial x} p_{00} = 0. \tag{3.26}$$

Therefore, we obtain

$$\langle T_{00} \rangle = -2\omega_T b^{-1} x, \tag{3.27}$$

$$p_{00} = p_{00}(y), \tag{3.28}$$

where the pressure $p_{00}(y)$ has to be determined with the aid of the boundary-layer solutions and where the arbitrary function of y, obtained on integrating (3.25), has been set equal to zero since equations (3.19) and (3.20) and the boundary conditions imply that $T_0(x) = -T_0(-x)$ and $p_0(x) = p_0(-x)$.

The boundary-layer equations are

$$\frac{\partial^2 \tilde{p}_{00}}{\partial \rho^2} = -b \cos\theta \frac{\partial}{\partial \rho} \langle \tilde{T}_{00} \rangle, \tag{3.29}$$

$$\frac{\partial^2 \langle \tilde{T}_{00} \rangle}{\partial \rho^2} = -\tfrac{1}{2} b \cos\theta \frac{\partial \tilde{p}_{00}}{\partial \rho}, \tag{3.30}$$

$$\frac{\partial^2 \tilde{\mathscr{T}}_{00}}{\partial \rho^2} = 0, \tag{3.31}$$

with the solutions

$$\tilde{p}_{00} = C_1(\theta) \frac{\cos\theta}{|\cos\theta|} 2^{\frac{1}{2}} \exp(-2^{-\frac{1}{2}} b |\cos\theta| \rho), \tag{3.32}$$

$$\langle \tilde{T}_{00} \rangle = C_1(\theta) \exp(-2^{-\frac{1}{2}} b |\cos\theta| \rho), \tag{3.33}$$

$$\tilde{\mathscr{T}}_{00} = 0. \tag{3.34}$$

We will first consider the problem where the wall temperature is specified and equal to zero. To satisfy the boundary conditions at $r = r_0$ we require that

$$\langle T_{00}(r = r_0) \rangle + \langle \tilde{T}_{00}(\rho = 0) \rangle = 0$$

and $$p_{00}(r=r_0)+\tilde{p}_{00}(\rho=0)=0. \tag{3.35}$$

These conditions determine $C_1(\theta)$ and $p_{00}(y)$ in the form

$$C_1(\theta)=2\omega_T b^{-1} r_0 \cos\theta$$

and $$p_{00}(r\sin\theta)=-2^{\frac{3}{2}}\omega_T b^{-1}(r_0^2-r^2\sin^2\theta)^{\frac{1}{2}}$$

or $$p_{00}(y)=-2^{\frac{3}{2}}\omega_T b^{-1}(r_0^2-y^2)^{\frac{1}{2}}. \tag{3.36}$$

The resulting velocity field in the interior is

$$u_{00}=-2^{\frac{1}{2}}\omega_T b^{-1} y (r_0^2-y^2)^{-\frac{1}{2}}. \tag{3.37}$$

We note that the lowest-order velocity is strictly parallel to the x axis. In addition, we can see that the interior vorticity has the same sign as the angular velocity of the top surface.

The above solutions are not valid in a small region of extent $r_0-r=O(\lambda^{-\frac{1}{3}})$ and $|\theta\pm\frac{1}{2}\pi|=O(\lambda^{-\frac{1}{6}})$ around the points $r=r_0$ and $\theta=\pm\frac{1}{2}\pi$ (i.e. $y=\pm r_0$). The interior velocity, which is singular at $y=\pm r_0$, becomes $O(\lambda^{-\frac{1}{3}})$ as this region is approached.

The remaining portion of the temperature field, $\mathscr{T}_{00}$, has yet to be determined. It satisfies the equation

$$\nabla^2\mathscr{T}_{00}=\tfrac{1}{2}b\,\partial[p_{01}(x,y)]/\partial x. \tag{3.38}$$

Since $\mathscr{T}_{00}$ does not participate directly in the boundary layers it must satisfy the boundary conditions on the side walls by itself. Also, it must take values at $z=0$ and $z=1$ such that $T_{00}(z=0)=T_{00}(z=1)=0$. Therefore, the boundary conditions on $\mathscr{T}_{00}$ are

$$\mathscr{T}_{00}(r=r_0)=0 \tag{3.39}$$

and $$\mathscr{T}_{00}(z=0)=\mathscr{T}_{00}(z=1)=-(\langle T_{00}\rangle+\langle\tilde{T}_{00}\rangle). \tag{3.40}$$

In addition, it must satisfy the condition

$$\int_0^1 \mathscr{T}_{00}\,dz=0. \tag{3.41}$$

It is easily shown that equation (3.38) and conditions (3.39)–(3.41) are sufficient to determine *both* $\mathscr{T}_{00}$ and p_{01x}.

The boundary condition (3.40) should also contain the z averaged solution for the temperature in the small regions around $r=r_0$ and $\theta=\pm\frac{1}{2}\pi$. However, a good approximation to the solution for $\mathscr{T}_{00}$ can be obtained by neglecting the boundary-layer corrections in condition (3.40) and approximating it by

$$\mathscr{T}_{00}(z=0)=\mathscr{T}_{00}(z=1)=-\langle T_{00}\rangle. \tag{3.42}$$

The resulting solution is

$$\mathscr{T}_{00}=4\omega_T b^{-1}\cos\theta\sum_{n=1}^{\infty}\frac{r_0 J_1(\alpha_n r/r_0)}{\alpha_n J_2(\alpha_n)}\left[\cosh\left[(z-\tfrac{1}{2})\frac{\alpha_n}{r_0}\right]-\frac{2r_0}{\alpha_n}\sinh\left(\frac{\alpha_n}{2r_0}\right)\right]$$
$$\times\left[\cosh\left(\frac{\alpha_n}{2r_0}\right)-\frac{2r_0}{\alpha_n}\sinh\left(\frac{\alpha_n}{2r_0}\right)\right]^{-1}, \tag{3.43}$$

where $J_1(\alpha_n) = 0$. With the use of the approximate condition (3.42), however, the accompanying formal representation of the solution for p_{01x} (which equals $4b^{-1}\mathscr{T}_{00z}(z = 1, x, y)$) is in the form of a divergent Fourier–Bessel series. The values of p_{01x} give the velocity components v_{01} and w_{01}. However, these components are $O(\lambda^{-1})$ and are therefore smaller than the primary velocity field u_{00}, which is $O(\lambda^{-\frac{1}{2}})$ and is given by (3.37). Consequently, the obtainment of an accurate representation of p_{01x} has not been pursued.

The main features of the flow for large λ have been found. We see that the interior velocity has fallen in magnitude from $O(1)$ to $O(\lambda^{-\frac{1}{2}})$ and, in addition, has become parallel to the x axis. The circulation closes in a boundary layer of thickness $\lambda^{-\frac{1}{2}}$, i.e. $E^{\frac{1}{4}}(\sigma S)^{-\frac{1}{2}}$, in which there are $O(1)$ velocities in the θ and z directions. Heat conduction is important in this boundary layer. However, it is strictly an inviscid layer since the viscous terms do not enter.

The basic relation for the interior z averaged temperature, equation (3.25), corresponds to the fact that, for $\lambda \gg 1$ the interior velocity has fallen in magnitude to $O(\lambda^{-\frac{1}{2}})$ and, consequently, an order 1 Ekman layer is not required on the bottom surface. The $O(E^{\frac{1}{2}})$ velocity $\mathbf{q}_1$, which is still required by the suction from the Ekman layer on the rotating top surface, must then satisfy $\mathbf{q}_1 \cdot \hat{\mathbf{n}}_B = 0$ at the container bottom and $\mathbf{q}_1 \cdot \hat{\mathbf{n}}_T = 2\omega_T$ at the top. Equation (3.25) results directly from these conditions, equation (3.10), and the z independence of w_1 and requires the existence of an $O(E^{\frac{1}{2}})$ temperature field in the interior. We see therefore that, as λ increases, the $O(1)$ interior flow crowds into boundary layers on the side walls where, in the heat equation, convection by the $O(1)$ vertical velocity balances the heat conduction resulting from the large gradients in temperature which are necessary to adjust the interior temperature field to the wall boundary condition.

With the solution for large λ determined, we find, on examination, that it remains a valid approximation for $\lambda < O(E^{-\frac{1}{2}})$, i.e. for $\sigma S < O(E)$. At $\sigma S = O(E)$ the thickness of the $\lambda^{-\frac{1}{2}}$ layer becomes $O(E^{\frac{1}{4}})$ and the viscous terms that are present in the $E^{\frac{1}{4}}$ side-wall layer have to be taken into account. Also, the velocity components v_{01} and w_{01} become $O(E^{\frac{1}{2}})$, which then coincides with the magnitude initially assumed for the other components, v_1 and w_1. It is also noteworthy that as $\sigma S \to O(E)$ the largest interior velocity component u_{00} falls in size to $O(E^{\frac{1}{4}})$.

4. Specified wall temperature: $\sigma S = O(E)$

Utilizing the information gained from the large λ solution of the last section we can proceed and study the flow for $\sigma S = O(E)$. We set $\mu = \sigma S/E$ and expand the interior variables as follows

$$\mathbf{q} = E^{\frac{1}{4}}u_0(y)\hat{\mathbf{i}} + E^{\frac{1}{2}}\mathbf{q}_1 + \ldots,$$
$$p = E^{\frac{1}{4}}p_0(y) + E^{\frac{1}{2}}p_1 + \ldots,$$
$$T = E^{\frac{1}{2}}T_0 + \ldots.$$

The equations for the $O(E^{\frac{1}{4}})$ and $O(E^{\frac{1}{2}})$ variables are

$$2u_0 = -p_{0y}, \tag{4.1}$$

$$\nabla \cdot \mathbf{q}_1 = 0, \tag{4.2}$$

$$2\hat{\mathbf{k}} \times \mathbf{q}_1 = -\nabla p_1 + T_0 \hat{\mathbf{k}}, \tag{4.3}$$

$$\mu \mathbf{q}_1 \cdot \hat{\mathbf{k}} = \nabla^2 T_0. \tag{4.4}$$

In particular, we find from (4.2) and (4.3) that

$$\partial w_1 / \partial z = 0. \tag{4.5}$$

The velocity component v_1 can again be written in the form given in (3.10).

An Ekman layer with an induced normal velocity of $O(E^{\frac{1}{2}})$ still exists on the top surface and the velocity boundary condition there is

$$\mathbf{q}_1 \cdot \hat{\mathbf{n}}_T = aw_1 - bv_1(z=1) = 2\omega_T. \tag{4.6}$$

On the bottom surface we require that

$$\mathbf{q}_1 \cdot \hat{\mathbf{n}}_B = -aw_1 + bv_1(z=0) = 0. \tag{4.7}$$

Adding equations (4.7) and (4.6) and using (3.10) and (4.5) we find that $\langle T_0 \rangle_x = -2\omega_T b^{-1}$. Therefore, we again obtain

$$\langle T_0 \rangle = -2\omega_T b^{-1} x, \tag{4.8}$$

where we have used, at this point, the result, found when the wall boundary conditions are applied to the interior and boundary-layer solutions, that the arbitrary function of y, obtained on integrating for $\langle T_0 \rangle$, has to be equal to zero to maintain $p_0 = p_0(y)$.

The largest side-wall layer now has a thickness which is $O(E^{\frac{1}{4}})$. We therefore introduce a stretched variable η, defined as

$$\eta = (r_0 - r) E^{-\frac{1}{4}}$$

and scale the boundary-layer *correction* variables as

$$\begin{aligned}
\bar{u} &= E^{\frac{1}{4}}\bar{u}_0 + E^{\frac{1}{2}}\bar{u}_1 + \ldots, \\
\bar{v} &= \bar{v}_0 + E^{\frac{1}{4}}\bar{v}_1 + \ldots, \\
\bar{w} &= \bar{w}_0 + E^{\frac{1}{4}}\bar{w}_1 + \ldots, \\
\bar{p} &= E^{\frac{1}{4}}\bar{p}_0 + E^{\frac{1}{2}}\bar{p}_1 + \ldots, \\
\bar{T} &= E^{\frac{1}{2}}\bar{T}_0 + \ldots.
\end{aligned}$$

The boundary-layer equations are

$$\begin{gathered}
-\bar{u}_{0\eta} + r_0^{-1}\bar{v}_{0\theta} + \bar{w}_{0z} = 0, \quad -\bar{u}_{1\eta} + r_0^{-1}\bar{v}_{1\theta} + \bar{w}_{1z} = 0. \\
-2\bar{v}_0 = \bar{p}_{0\eta}, \quad -2\bar{v}_1 = \bar{p}_{1\eta}, \\
2\bar{u}_0 = -r_0^{-1}\bar{p}_{0\theta}, \quad 2\bar{u}_1 = -r_0^{-1}\bar{p}_{1\theta} + \bar{v}_{0\eta\eta}, \\
\bar{p}_{0z} = 0, \quad 0 = -\bar{p}_{1z} + \bar{T}_0 + \bar{w}_{0\eta\eta}, \\
\mu\bar{w}_0 = \bar{T}_{0\eta\eta}.
\end{gathered} \tag{4.9}$$

It follows that
$$\bar{w}_{0z} = \bar{u}_{0z} = \bar{v}_{0z} = \bar{T}_{0z} = 0,$$

$$\overline{w}_{1z} = \tfrac{1}{2}\overline{v}_{0\eta\eta\eta}, \tag{4.10}$$

$$\overline{v}_{1z} = -\tfrac{1}{2}(\overline{w}_{0\eta\eta\eta} + \overline{T}_{0\eta}). \tag{4.11}$$

In addition, from the boundary conditions $\overline{\mathbf{q}}_0 \cdot \hat{\mathbf{n}}_T = \overline{\mathbf{q}}_0 \cdot \hat{\mathbf{n}}_B = 0$ we obtain

$$a\overline{w}_0 = b\cos\theta\overline{v}_0. \tag{4.12}$$

The Ekman layer conditions at the top and bottom surfaces, within the $E^{\frac{1}{4}}$ layer, can be expressed as

$$a\overline{w}_1(z=1) - b\cos\theta\overline{v}_1(z=1) - b\sin\theta\overline{u}_0 = \tfrac{1}{2}a^{-\frac{1}{2}}(b\cos\theta\overline{w}_{0\eta} + a\overline{v}_{0\eta}), \tag{4.13}$$

$$a\overline{w}_1(z=0) - b\cos\theta\overline{v}_1(z=0) - b\sin\theta\overline{u}_0 = -\tfrac{1}{2}a^{-\frac{1}{2}}(b\cos\theta\overline{w}_{0\eta} + a\overline{v}_{0\eta}). \tag{4.14}$$

Subtracting (4.14) from (4.13), using (4.12), and substituting the results of integrating (4.10) and (4.11) with respect to z, we obtain

$$\frac{\partial^3\overline{v}_0}{\partial\eta^3}[a + (b^2/a)\cos^2\theta] + b\cos\theta\frac{\partial\overline{T}_0}{\partial\eta} = 2a^{-\frac{1}{2}}[a + (b^2/a)\cos^2\theta]\frac{\partial\overline{v}_0}{\partial\eta}. \tag{4.15}$$

It is convenient to work with the variables $\overline{p}_0$ and $\overline{T}_0$. Therefore, using the relation $\overline{v}_0 = -\frac{1}{2}\overline{p}_{0\eta}$ and (4.12) and, in addition, considering $b \ll 1$, we find that (4.9) and (4.15) reduce to the two governing equations for the boundary-layer variables.

$$\frac{\partial^4\overline{p}_0}{\partial\eta^4} - 2\frac{\partial^2\overline{p}_0}{\partial\eta^2} - 2b\cos\theta\frac{\partial\overline{T}_0}{\partial\eta} = 0, \tag{4.16}$$

$$\frac{\partial^2\overline{T}_0}{\partial\eta^2} + \tfrac{1}{2}\mu b\cos\theta\frac{\partial\overline{p}_0}{\partial\eta} = 0. \tag{4.17}$$

The solutions to (4.16) and (4.17) can be written as

$$\overline{T}_0 = D_1(\theta)\,e^{-s_1\eta} + D_2(\theta)\,e^{-s_2\eta} \tag{4.18}$$

and

$$\overline{p}_0 = 2(s_1D_1e^{-s_1\eta} + s_2D_2e^{-s_2\eta})(\mu b\cos\theta)^{-1}, \tag{4.19}$$

where s_1 and s_2 are the two roots, with positive real parts, of

$$s^2 = 1 \pm (1 - \mu b^2\cos^2\theta)^{\frac{1}{2}}. \tag{4.20}$$

The appropriate boundary conditions to be applied at the wall are

$$\overline{p}_{0\eta}(\eta = 0) = 0, \tag{4.21}$$

$$p_0(r = r_0) + \overline{p}_0(\eta = 0) = 0, \tag{4.22}$$

$$\langle T_0(r = r_0)\rangle + \overline{T}_0(\eta = 0) = 0, \tag{4.23}$$

where (4.21) comes from the requirement that $\overline{v}_0(\eta = 0) = 0$. Applying these conditions we find

$$D_2 = -(s_1/s_2)^2D_1.$$

$$D_1 = 2\omega_Tb^{-1}r_0\cos\theta(1 - (s_1/s_2)^2)^{-1},$$

and

$$p_0(r_0\sin\theta) = -\frac{4s_1s_2\omega_Tr_0}{\mu b^2(s_1+s_2)}. \tag{4.24}$$

Let us first examine the interior solution for the case $\mu \ll 1$. Approximations to the two roots can be found from (4.20) and are

$$s_1 \approx 2^{\frac{1}{2}}, \quad s_2 \approx \mu^{\frac{1}{2}} 2^{-\frac{1}{2}} b|\cos\theta|. \tag{4.25}$$

We recognize s_1 as the usual $E^{\frac{1}{4}}$ layer root and, taking into account the boundary-layer scaling, we recognize s_2 as the $\lambda^{-\frac{1}{2}}$ layer root obtained in §3. In this case p_0 simply has a small $O(\mu^{\frac{1}{2}})$ correction to the expression given in (3.36).

When μ increases to the value $\mu = b^{-2}$ the two real roots coalesce at the points $\theta = 0, \pi$. For larger values of μ the roots are complex conjugates at values of θ such that $|\cos\theta| > (\mu b^2)^{-\frac{1}{2}}$. For $\mu \gg 1$ and for $|\cos\theta| \gg (\mu b^2)^{-\frac{1}{2}}$ the roots are given approximately by

$$s_{1,2} \approx (1 \pm i)(\mu^{\frac{1}{2}} b|\cos\theta|)^{\frac{1}{2}} 2^{-\frac{1}{2}}. \tag{4.26}$$

Using approximation (4.26) in (4.24) we find that the interior pressure is determined in the form

$$p_0(y) = -2^{\frac{3}{2}} \frac{\omega_T r_0^{\frac{1}{2}}}{b^{\frac{3}{2}} \mu^{\frac{3}{4}}} (r_0^2 - y^2)^{\frac{1}{4}}. \tag{4.27}$$

The resulting velocity field is

$$u_0(y) = -\frac{\omega_T r_0^{\frac{1}{2}} y}{2^{\frac{1}{2}} b^{\frac{3}{2}} \mu^{\frac{3}{4}} (r_0^2 - y^2)^{\frac{3}{4}}}, \tag{4.28}$$

where again the solutions are not valid in small regions around the points $y = \pm r_0$. The lowest-order velocity (4.27) is still parallel to the x axis but the explicit dependence on y has changed.

The remaining portion of the temperature field $\mathscr{T}_0 = T_0 - (\langle T_0 \rangle + \overline{T}_0)$ and the velocity component w_1 can be obtained in the same manner as indicated for the variables $\mathscr{T}_{00}$ and p_{01x} in §3. In fact, $\mathscr{T}_0$ has the same approximate value as that given for $\mathscr{T}_{00}$ in (3.43) since the z averaged interior temperature is the same in both cases. The velocity component v_1 is given by the expression

$$v_1 = \frac{1}{b} w_1(x, y) + \tfrac{1}{2} \frac{\partial}{\partial x} \int_0^z T_0 \, dz. \tag{4.29}$$

For $\mu \gg 1$, $w_1 = O(\mu^{-1})$ and therefore v_1 is approximately equal to the second term in the above equation.

The solutions for large μ are valid for $\mu < O(E^{-\frac{1}{3}})$ or $\sigma S < O(E^{\frac{2}{3}})$. At the point $\sigma S = O(E^{\frac{2}{3}})$ the lowest order interior pressure field has fallen in magnitude to $O(E^{\frac{1}{2}})$ and this invalidates our prior assumptions. In addition, the boundary layer thickness varies as $E^{\frac{1}{4}} \mu^{-\frac{1}{4}}$, i.e. as $E^{\frac{1}{2}} (\sigma S)^{-\frac{1}{4}}$, which, at $\sigma S = O(E^{\frac{2}{3}})$ becomes $O(E^{\frac{1}{3}})$, necessitating the inclusion of the additional viscous terms present in the $E^{\frac{1}{3}}$ layer. We note that the stratification $\sigma S = O(E^{\frac{2}{3}})$ is the critical value for the side-wall layers found by Barcilon & Pedlosky (1967*b*). At this critical stratification the magnitude of the interior velocity field in the 'doubly sliced' cylinder has fallen to $O(E^{\frac{1}{2}})$ while the largest velocities in the side-wall layers are $O(E^{\frac{1}{6}})$. The boundary-layer scale of $E^{\frac{1}{2}} (\sigma S)^{-\frac{1}{4}}$ was also found by Barcilon & Pedlosky and is that of the buoyancy layer.

5. Adiabatic wall: $\sigma S = O(E^{\frac{3}{2}})$

Let us consider now the case where the boundary condition on the temperature at the container surface is one of no heat flux, i.e. where

$$\hat{\mathbf{n}} \cdot \nabla T = 0 \tag{5.1}$$

on the boundaries. We return to the developments in §3, where $\sigma S = O(E^{\frac{3}{2}})$, and solve the problem posed by (3.19) and (3.20) with boundary conditions (3.8) and (5.1). We are again interested in the solution for $\lambda \gg 1$ and expand the variables in the same manner as given in (3.23) and (3.24).

On the top and bottom surfaces the boundary condition (5.1) becomes, with the approximation $b \ll 1$, $T_{00z}(z = 1) = T_{00z}(z = 0) = 0$. As a consequence, we find $\mathscr{T}_{00} \equiv 0$ which, of course, implies $T_{00} \equiv \langle T_{00} \rangle$ and for convenience we drop the average symbols.

The interior temperature T_{00} and pressure p_{00} are again given by the expressions (3.27) and (3.28). The boundary-layer equations are the same as before and the solutions are given by (3.32) and (3.33). The conditions to be applied at the wall are (3.35) and

$$\tilde{T}_{00\rho}(\rho = 0) = 0, \tag{5.2}$$

$$T_{00r}(r = r_0) - \tilde{T}_{01\rho}(\rho = 0) = 0. \tag{5.3}$$

It follows from (5.2) that $\tilde{T}_{00} \equiv 0$. This in turn implies $\tilde{p}_{00} \equiv 0$. Equations (3.28) and (3.35) then require that $p_{00} \equiv 0$. Therefore, no boundary-layer corrections are needed at this order and, consequently, no interior pressure or velocity field of order $\lambda^{-\frac{1}{2}}$ is present. With the adiabatic wall condition, only an adjustment in slope of the interior temperature field is required in the boundary layer. This results in smaller, by $O(\lambda^{-\frac{1}{2}})$, boundary-layer temperature gradients compared with the case where the value of the temperature itself has to be adjusted. As a result, we find that, with an adiabatic wall, the velocities in the boundary layer, and hence in the interior, are also smaller by at least $O(\lambda^{-\frac{1}{2}})$.

The boundary-layer equations and solutions at the next order, for $\tilde{T}_{01}$ and $\tilde{p}_{01}$ are identical with those for $\tilde{T}_{00}$ and $\tilde{p}_{00}$. In addition, the interior variables must satisfy

$$\nabla^2 T_{00} = \tfrac{1}{2} b p_{01x}$$

from which, with (3.27), we obtain $p_{01x} = 0$ and consequently that $p_{01} = p_{01}(y)$. The boundary condition on the pressure is

$$p_{01}(r = r_0) + \tilde{p}_{01}(\rho = 0) = 0. \tag{5.4}$$

Applying conditions (5.3) and (5.4) we find that the solutions are

$$\tilde{T}_{01} = 2^{\frac{3}{2}} \frac{\omega_T \cos\theta}{b^2 |\cos\theta|} \exp(-2^{-\frac{1}{2}} b |\cos\theta| \rho),$$

$$\tilde{p}_{01} = 4\omega_T b^{-2} \exp(-2^{-\frac{1}{2}} b |\cos\theta| \rho),$$

$$p_{01} = -4\omega_T b^{-2}.$$

The largest velocities in the boundary layer are $O(\lambda^{-\frac{1}{2}})$. However, since p_{01} is a constant there is, in this case, no interior velocity field of order λ^{-1}. If we attempt to calculate the lowest-order non-zero interior velocity field by applying the same procedure to find higher order terms, we encounter inconsistencies. Apparently, a consideration of the solutions in the regions near the points $r = r_0, \theta = \pm\frac{1}{2}\pi$ has to be included and the results govern the form of the expansion to higher order. An attempt at this extension is met with analytical difficulties and has not been completed. We note, however, that for a more general form of the driving motion of the top surface the lowest-order interior pressure p_{01} is not constant and a non-zero interior velocity field of order λ^{-1} exists. For example, with a rotation of the top surface at a variable velocity of $v'_T = (r'/r_0)^3$, where v'_T and r' refer to polar co-ordinates in the plane of the top surface, we find

$$p_{01} = -16\omega_T r_0^{-2} b^{-2}(x^2 + 2y^2 - \tfrac{1}{2}r_0^2).$$

Note that here the lowest-order velocity field is not parallel to the x axis.

In general, therefore, for an adiabatic wall and for $\lambda \gg 1$ we conclude that the interior velocity field is $O(\lambda^{-1})$ or smaller. The large λ solutions are again valid for $\sigma S < O(E)$ and the above results indicate that for $\sigma S = O(E)$ the interior velocity field is $O(E^{\frac{1}{2}})$.

We should add the comment that if solutions are sought for stratifications with $\sigma S > O(E)$ the solution for the Ekman layer on the top surface results in a non-zero heat-flux boundary condition on an interior temperature field of magnitude $O(\sigma S/E^{\frac{1}{2}})$. This is greater than $O(E^{\frac{1}{2}})$ for $\sigma S > O(E)$ and can be expected to cause additional complications in the adjustment of the flow field.

6. Conclusions

We have considered the effects of a weak stratification on the steady, linear motion of a fluid in a rotating, 'doubly sliced' cylinder. For a surface-boundary condition of zero perturbation temperature we found that, as the stratification increased from zero, the characteristics of the flow of a homogeneous fluid were greatly altered before the critical stratification $\sigma S = O(E)$ was reached. In the region $E^{\frac{3}{2}} < \sigma S < E$ the largest component of the interior velocity field had magnitude $O(E^{\frac{3}{4}}(\sigma S)^{-\frac{1}{2}})$ and was parallel with the x axis. An inviscid, but heat-conducting boundary layer of thickness $E^{\frac{3}{4}}(\sigma S)^{-\frac{1}{2}}$ existed on the side wall. As the stratification increased to $O(E^{\frac{2}{3}})$ the interior velocity fell in magnitude to $O(E^{\frac{1}{2}})$. With a boundary condition of zero heat flux we found that, in general, the interior velocity field fell to $O(E^{\frac{1}{2}})$ for stratifications approaching $O(E)$.

These results are very different from those found by Barcilon & Pedlosky (1967*b*) for the steady motion in a cylinder with flat top and bottom surfaces. In that case, the largest component of the interior velocity field remains identical to the result for a homogeneous fluid, given by (3.22), for stratifications approaching $O(E^{\frac{1}{2}})$. Therefore we can see that, according to the linear theory, a small change in geometry that leaves the characteristics of the flow of a homogeneous fluid unaltered can result in a substantial modification of the flow field when a weak stratification is present and interacts with the geostrophic flow.

This research was supported by the Atmospheric Sciences Section, National Science Foundation, under grant GA-1019.

REFERENCES

ALLEN, J. S. 1968 Some aspects of the inviscid motion of a contained rotating and stratified fluid. *Notes on the* 1968 *Summer Program in Geophysical Fluid Dynamics, Woods Hole Oceanographic Institution*, ref. no. 68–72, vol. 2, 1–19.

BARCILON, V. & PEDLOSKY, J. 1967*a* Linear theory of rotating stratified fluid motions. *J. Fluid Mech.* **29**, 1–16.

BARCILON, V. & PEDLOSKY, J. 1967*b* A unified linear theory of homogeneous and stratified rotating fluids. *J. Fluid Mech.* **29**, 609–621.

COLE, J. D. 1968 *Perturbation Methods in Applied Mathematics.* New York: Blaisdell.

GREENSPAN, H. P. 1968 *The Theory of Rotating Fluids*. Cambridge University Press.